PRINCETON COS CON

Problem 7: Lattice Lights

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(Physicists and electrical engineers: please don't skewer me for my questionable background in electromagnetism. I apologize in advance if the physics behind this problem doesn't make any sense.)

Consider the lattice formed by the integer points (a, b, c), where $0 \leq a, b, c < N$ (an integer point is one where $a, b, c \in \mathbb{Z}$). At every integer point in this lattice, we have a small circuit consisting of some diodes and a light. We also have $3N$ switches, each of which can turn on a potential: N switches to turn on a potential V_x^k that is nonzero only along the plane $x = k$, N switches to turn on a potential V_y^k that is only nonzero along the plane $y = k$, and N switches to turn on a potential V_z^k that is only nonzero along the plane $z = k$ (for each of these, k ranges from 0 to $N - 1$, inclusive). The effect of this setup is to turn on the light at point (a, b, c) if and only if all three potentials V_x^a, V_y^b, and V_z^c are turned on.

Every integer point is between 0 and $3(N - 1)$ hops from the origin. We are interested in collecting statistics about how far shining lights are from the origin. The problems we wish to solve are as follows:

Input: We are given an integer $N \geq 3$ denoting the size of the lattice, three lists A_x, A_y, and A_z, each of length between 1 and N, inclusive. A_x is guaranteed to be a subset of $\{0, 1, \cdots, N - 1\}$ and lets us know which of the V_x^k are turned on, i.e. $k \in A_x$ if and only if V_x^k is turned on. The structure of A_y and A_z are similar, and let us know which of the V_y^k and V_z^k are turned on. (All indexing starts at zero.) We are also given a number $C \leq \lceil \log_2 N \rceil$ (whose purpose will be explained when describing the desired output of problem B).

Problem A: Output the average (taxicab) distance that shining lights are at, which is given by

$$\mu = \frac{1}{M} \sum_{(a,b,c)} d(a, b, c) \cdot \mathbb{1}_{V_x^a \text{ is on, } V_y^b \text{ is on, } V_z^c \text{ is on}} = \frac{1}{M} \sum_{(a,b,c)} d(a, b, c) \cdot \mathbb{1}_{a \in A_x} \cdot \mathbb{1}_{b \in A_y} \cdot \mathbb{1}_{c \in A_z}$$

where $d(a, b, c) = |a| + |b| + |c|$ is the taxicab distance from the origin, M is the total number of shining lights, and the sum is taken over all points in the lattice, i.e. $\sum_{(a,b,c)} = \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1}$.

Problem B: Output the C distances which have the most lights on in order from most to least (recall again that all distances are measured as the taxicab distance). If there are ties, you may output any set of C distances such that none of the remaining $3(N - 1) - C$ distances have a strictly greater number of shining lights than each of the elements in your set.

(i) (2 points) Given some distance $0 \leq D \leq 3(N-1)$, what is the maximum number of shining lights at distance D? Give your answer in big-theta notation as a function of N. What does this imply, if anything, about the worst case running time about an algorithm that solves problem A? What does this imply, if anything, about the worst case running time about an algorithm that solves problem B?

(ii) (3 points) Describe an algorithm to solve problem A. What is its worst-case running time, in terms of N? In addition, give a brief justification of its correctness. If your algorithm is not correct, then we cannot give credit. If it is correct, then your algorithm will be judged on its worst-case running time.*

(iii) (10 points) Describe an algorithm to solve problem B. What is its worst-case running time, in terms of N and C? In addition, give a brief justification of its correctness. If your algorithm is not correct, then we cannot give credit. If your algorithm is correct, it will be judged on its worst-case running time.†

*The worst-case running time is only a function of N and is defined as the maximum amount of operations needed to solve any instance of the problem for a fixed N.

†Here, the worst-case running time is a function of N and C and is defined as the maximum amount of operations needed to solve any instance of the problem for a fixed N and C.

See the next page for an example and illustration.

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Example

Suppose we are given the following input:

$$N = 3, A_x = [0, 1, 2],\ A_y = [1, 2],\ A_z = [1],\ C = 2$$

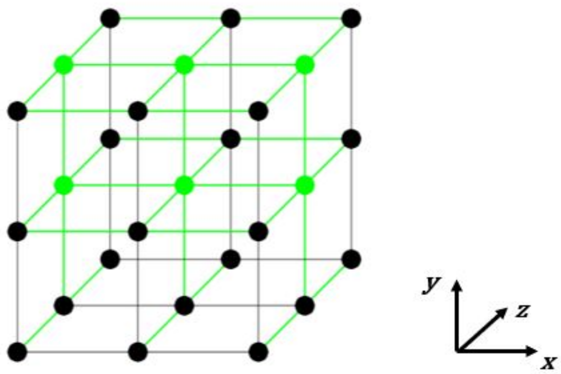
The points which have the lights on are

$$(0, 1, 1), (0, 2, 1), (1, 1, 1), (1, 2, 1), (2, 1, 1), (2, 2, 1)$$

and thus the average distance shining lights are at is

$$\frac{1}{6}(d(0, 1, 1) + d(0, 2, 1) + d(1, 1, 1) + d(1, 2, 1) + d(2, 1, 1) + d(2, 2, 1)) = \frac{7}{2}$$

Note that there are two shining lights at distance 4 and two shining lights at distance 3. So a correct answer to problem B for this input must output a list containing 3 and 4 (in either order).



An illustration of the example. Note that the only lights that are on are the ones which have three green perpendicular wires coming into them. Clarification on how the wires are colored: if V_x^k is turned on, then all wires going in and out of the page on the plane $x = k$ are green; if V_y^k is turned on, then all horizontal wires on the plane $y = k$ are green; if V_z^k is turned on, then all vertical wires on the plane $z = k$ are green.